Algebra Univers. 76 (2016) 493–495
DOI 10.1007/s00012-016-0412-1
Published online October 14, 2016

Algebra Universalis

Modularity for upper continuous and strongly atomic lattices

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Dedicated to Professor Jan Zygmunt on the occasion of his 70th birthday

Abstract. This paper extends a result of Crawley and Dilworth on upper continuous and strongly atomic modular lattices.

Standard lattice-theoretic notions such as modularity, covering relation, ascending and descending chain conditions, etc. can be found in [4]. For the reader's convenience, let us recall some definitions and facts: a lattice L is said to be *upper continuous* if it is a complete lattice and the following condition is satisfied for any $x \in L$ and for any chain $C \subseteq L$:

$$x \wedge \bigvee C = \bigvee \{x \wedge c \colon c \in C\}. \tag{UC}$$

A lattice L is called *strongly atomic* if

$$(\forall x, y \in L)\big(x < y \Rightarrow (\exists z \in L)(x \prec z \leq y)\big). \tag{SA}$$

It is a trivial fact that any finite lattice is upper continuous and strongly atomic. Moreover, let us list five important consequences of modularity, the so-called *isomorphism theorem*, *upper* and *lower semimodularity laws*, and *upper* and *lower Birkhoff's conditions*:

$$(\forall x, y \in L)\big([x \wedge y, x] \cong [y, x \vee y]\big), \tag{Iso}$$

$$(\forall x, y \in L)(x \wedge y \prec x \Rightarrow y \prec x \vee y), \tag{Sm}$$

$$(\forall x, y \in L)(y \prec x \vee y \Rightarrow x \wedge y \prec x), \tag{Sm*}$$

$$(\forall x, y \in L)(x \wedge y \prec x, y \Rightarrow x, y \prec x \vee y), \tag{Bi}$$

$$(\forall x, y \in L)(x, y \prec x \vee y \Rightarrow x \wedge y \prec x, y). \tag{Bi*}$$

Clearly, (Iso) implies (Sm) and (Sm*), (Sm) implies (Bi), and (Sm*) implies (Bi*), but the converses of these implications do not hold. However, M. Ward proved the following theorem.

Proposition 1 ([7]). *If L satisfies ascending or descending chain condition, then* (Iso) *implies the modularity of L.*

Presented by M. Ploscica.
Received March 24, 2015; accepted in final form October 19, 2015.
2010 *Mathematics Subject Classification*: Primary: 06D05; Secondary: 06B05.
Key words and phrases: modular lattice, Birkhoff's conditions, upper semimodular lattice, strongly atomic lattice, upper continuous lattice.

R. P. Dilworth and P. Crawley extended Ward's result and specified the relation between (Sm) and (Bi).

Proposition 2 ([2, Theorem 3.6]). *If L is an upper continuous and strongly atomic lattice, then conditions* (Sm) *and* (Sm*) *imply the modularity of L.*

Proposition 3 ([2, Theorem 3.7]). *If L is an upper continuous and strongly atomic lattice, then condition* (Bi) *implies* (Sm).

Although the authors of the preceding theorems formulate the assertions for compactly generated (i.e., algebraic) and strongly atomic lattices, their proofs use upper continuity and strong atomicity only (cf. [6, p. 39]). Note also that upper continuity and strong atomicity do not imply algebraicity in general (see [5, p. 338]).

Applying the technique developed by Dilworth and Crawley in [2, 3], we prove a strengthening of Proposition 2.

Proposition 4. *If L is an upper continuous and strongly atomic lattice, then conditions* (Bi) *and* (Bi*) *imply the modularity of L.*

Proof. According to Propositions 2 and 3, it suffices to show that L satisfies (Sm*). Suppose to the contrary that for some $s, t \in L$, $t \prec s \vee t$, but $s \wedge t \not\prec s$. Put $v = s \wedge t$ and $u = s \vee t$. By (SA), there are $a, b \in L$ such that $v \prec b \prec a \leq s$. Hence, $\{v, a, b, t, u\}$ forms a pentagon.

Consider the set T of all ordered triples $(x, y, z) \in L^3$ such that

$$a \leq x \leq u,\ b \leq y,\ v \leq z,\ t \vee y = u,\ t \wedge x = z,\ z \prec y \prec x,\ a \wedge y = b, \quad (1)$$

partially ordered by the relation

$$(x, y, z) \leq (x', y', z') \quad \Leftrightarrow \quad x \leq x' \text{ and } y \leq y' \text{ and } z \leq z'.$$

Since $(a, b, v) \in T$, T is nonempty. As a consequence of (1), observe that $y = b \vee z$ and $x = a \vee y$, provided that $(x, y, z) \in T$. Assume that the set $\{(x_i, y_i, z_i) \colon i \in I\} \subseteq T$ is a chain. Then (Sm) and (UC) yield that $(\bigvee x_i, \bigvee y_i, \bigvee z_i)$ belongs to T and it is an upper bound of this chain. According to the Kuratowski–Zorn Lemma, there is a maximal triple $(a_0, b_0, v_0) \in T$ such that $\{v_0, a_0, b_0, t, u\}$ forms a pentagon.

By (Bi), $v_0 \not\prec t$; therefore, (SA) guarantees the existence of $v_1 \in L$ such that $v_0 \prec v_1 < t$. Put $a_1 = a_0 \vee v_1$ and $b_1 = b_0 \vee v_1$. In order to get a contradiction, we will show that $(a_1, b_1, v_1) \in T$.

Of course, we have $a \leq a_1 \leq u$, $b \leq b_1$, $v \leq v_1$, and $t \vee b_1 = u$. Applying (Bi), one can prove that

$$v_1 \prec b_1, \quad b_0 \prec b_1, \quad b_1 \prec a_1, \quad a_0 \prec a_1, \quad (2)$$

which justifies the condition $v_1 \prec b_1 \prec a_1$, and moreover,

$$b_1 \wedge t = v_1, \quad a_0 \vee (t \wedge a_1) = a_1, \quad a \vee b_0 = a_0, \quad a_0 \vee b_1 = a_1. \quad (3)$$

Now we will prove that $a \wedge b_1 = b$. Since $b \leq a \wedge b_1 \leq a$ and $b \prec a$, we have $a \wedge b_1 = b$ or $a \wedge b_1 = a$. However, the latter alternative is impossible because together with (3), it gives $a_1 = b_1$, which contradicts (2).

Lastly, we show that $t \wedge a_1 = v_1$. If, to the contrary, $t \wedge a_1 \neq v_1$, then by (2), (3), and (Sm), we get $t \wedge a_1 \prec a_1$; thus, $a_0, t \wedge a_1 \prec a_1 = a_0 \vee (t \wedge a_1)$. Now it suffices to employ (Bi*) to get $v_0 \prec a_0$, which is impossible.

Therefore, since the triple (a_1, b_1, v_1) fulfills the conditions listed in (1), so $(a_1, b_1, v_1) \in T$, a contradiction to the maximality of (a_0, b_0, v_0). Finally, $s \wedge t \prec s$, which completes the proof. $\square$

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